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NUMERICAL SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS USING OPERATIONAL MATRIX METHOD

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Abstract
This paper investigates a novel numerical scheme based on the operational matrix method and the second kind of Chebyshev polynomials to numerically solve some non-linear differential equations (NODEs). The proposed numerical method is given the truncated second-kind Chebyshev polynomial series solution of the relevant NODEs. The operational matrix method is applied to set a non-linear algebraic system to find such unknown series coefficients. Additionally, some inspections are revealed to analyze the convergency and error bounds of the method. Finally, eight problems are solved by the proposed method to show the effectiveness and trueness of the method.

Key Words: Operational matrix method, Non-linear differential equations, Riccati equation, Duffing equation, Van der Pol equation.

1. INTRODUCTION

We will take the following high-order non-linear differential equations [1, 2]:

$$u^p(x) = \sum_{q=0}^{n} f_q(x)u^{(q)}(x) + g(x) \tag{1}$$

with initial conditions

$$u(0) = a_0, u'(0) = a_1, \dots, u^{n-1}(0) = a_{n-1} \tag{2}$$

where $f_q(x)$ and $g(x)$ are continuous functions and defined on $-1 \le x \le 1$, for $i = 0, 1, 2, \dots, n-1$, a_i is proper constant.

Non-linear differential equations (NODEs) are critical in modelling complex phenomena where linear approximations are insufficient. From the behaviour of fluids to the dynamics of populations, from the chaos of weather systems to the oscillations in chemical reactions, NODEs are essential tools for understanding and predicting the behaviour of real-world systems. They provide insights into stability, pattern formation, bifurcations and chaos, enabling advancements in science, engineering, and technology. While NODEs have spectacular attention in making models and investigating their numerical solutions, solutions of NODEs cannot be obtained directly from classical or even numerical methods. Nevertheless, professional researchers have exhibited some efficient method to solve NODEs numerically. Some of them applied in science contexts can be summarized as follows: In [3], the Boole collocation method is to be developed for the solution of cubic non-linear differential equations based on Boole polynomials. This method sets a matrix-vector form of the given equation that transforms non-linear algebraic systems through collocation points. In [4], the Chelyshkov matrix method was proposed to solve the nonlinear fractional delay differential equation. The fractional term is defined in the Caputo sense, and to find its solution, the unknown solution is discretized using a truncated series based on orthogonal Chelyshkov functions. In [5], a similarity transformation method is attempted for the bulk flow, micro spin flow and heat transfer phenomenon for micropolar fluids dynamics through Darcy porous medium. In this study, Simpson's Rule and successive over-relaxation methods are implemented to NODEs to get numerical results. In [6], the Lucas series and collocation method were used to obtain the numerical solution of nonlinear differential equations with variable delays. The procedure of the numeric method converts the non-linear equation to a matrix-vector form that appears as a system of non-linear algebraic equations with unknown coefficients of the Lucas series. Moreover, some basic numerical schemes can be given hybrid classical-quantum algorithms [7], Walsh functions [8], Gamma function method [9], Haar wavelet method [10], and Chebshev series method [11].

The operational matrix method (OMM) has recently been paid attention to discovering the numerical solution of crucial equations. It is particularly effective in problems involving polynomials, orthogonal polynomials, and functions that can be expressed as a basis of functions. The method's central policy is forcing zero of the residual functions [12, 13]. The method typically transforms the given differential or integral equation into a system of algebraic equations using an operational matrix representation of differential or integral operators. This method is widely applied in control theory, signal processing, and mathematical physics. The main advantage of OMM is that the given equation turns into a simple matrix multiplication, which is computationally efficient. Some authors have used OMM to obtain numerical solutions for different equations. For example, in [14], a Vieta Lucas OMM was applied moreover numerical solutions systems of fractional differential equations were obtained. In [15], shifted Chebyshev polynomials of the third kind are considered for multi-term fractional differential equations. In [16], the Chebyshev wavelets operational matrix method has produced many differential equations. Moreover, the fractional differential equation by the Tau-Collocation method [17], Chebyshev operational matrix method to solve the Lane-Emden equation [18], Fredholm-Volterra integro-differential equation [19], the second kind of Chebyshev operational matrix method to solve the Lane-Emden equation [20], an operational matrix method for solving the fractional-order differential equations [21], optimal control problems [22], the Euler-Poisson-Darboux equation [23] can be given.

2. THE SECOND KIND OF CHEBYSHEV POLYNOMIALS AND THEIR OPERATIONAL MATRICES

The analytic equation of the second kind of Chebyshev polynomials can be given by [24]

$$U_n(x) = \sum_{s=0}^{\lfloor n/2 \rfloor} (-1)^s 2^{n-2s} \binom{n-s}{s} x^{n-2s} \tag{3}$$

where $\lfloor x \rfloor$ is the floor function of x. All second-kind Chebyshev polynomials can be produced following relation [24]

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \ldots \tag{4}$$

which, together with the initial conditions

$$U_0(x) = 1, \quad U_1(x) = 2x$$

In the literature, many researchers are investigating relations between matrices and polynomials such as Pell-Lucas polynomial and Fibonacci polynomial. The second-kind Chebyshev polynomial can also be expressed by

$$U^n = \begin{bmatrix} U_n(x) & -U_{n-1}(x) \\ U_{n-1}(x) & -U_{n-2}(x) \end{bmatrix} \tag{5}$$

where

$$U = \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} U_1(x) & -U_0(x) \\ U_0(x) & 0 \end{bmatrix} \tag{6}$$

and the matrix U is called the generator matrix of the second kind of Chebyshev polynomials. That is, when we take the powers of the generator matrix, the component of the matrix U^n gives the second kind of Chebyshev polynomial.

For example, for $n = 4$ we have U_4, U_3 and U_2 as follows:

$$U^n = \begin{bmatrix} U_4(x) & -U_3(x) \\ U_3(x) & -U_2(x) \end{bmatrix} = \begin{bmatrix} 16x^4 - 12x^2 + 1 & -(8x^3 - 4x) \\ 8x^3 - 4x & -(4x^2 - 1) \end{bmatrix}$$

Moreover, the second kind of Chebyshev polynomials are orthogonal on $[-1, 1]$ concerning the weight function $\omega(x) = (1 - x^2)^{1/2}$

$$\int_{-1}^{1} U_r(x)U_s(x)\omega(x)dx = \begin{cases} \pi, & r = s = 0 \\ \frac{\pi}{2}, & r = s \neq 0 \\ 0, & r \neq s \end{cases} \quad (7)$$

Since the second kind of Chebyshev polynomials are orthogonal polynomials, they use the function approximation [24]. If we choose any functions $w(x) \in L^2[-1, 1]$, a second kind Chebyshev polynomial representation can be built as

$$w(x) = \sum_{k=0}^{\infty} c_i U_i(x) \quad (8)$$

where

$$c_i = \frac{<w(x),U_i(x)>}{<U_i(x),U_i(x)>} = \sigma_i \int_{-1}^{1} w(x)U_i(x)dx \quad (9)$$

for $i = 0$, $\sigma_i = \pi$ and for others i values $\sigma_i = \pi/2$. If one can be approximated to $w(x)$, it is enough to take finite terms of Eq.(8) which is called the truncated first (N+1) second kind of Chebyshev polynomials w_N and its matrix form as:

$$w_N(x) = \sum_{k=0}^{N} c_i U_i(x) = CH(x) \quad (10)$$

where

$$C = [c_0 \;\; c_1 \;\; c_2 \;\; \cdots \;\; c_N]$$

$$H(x) = [U_0(x) \; U_1(x) \; U_2(x) \; \cdots \; U_N(x)]^T$$

We want to build the operational matrix method with terms of the standard bases by using Eq. (3). For this purpose, it can be written the first and second derivatives of H(x) as

$$H(x) = E^{-1}Y(x) \quad (11)$$

$$H'^{(x)} = E^{-1}TY(x) \quad (12)$$

and for any positive q integer

$$H^{(q)}(x) = E^{-1}T^qY(x) \quad (13)$$

where for odd N ,

$$E=\begin{bmatrix} \frac{1}{2^0}\binom{0}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2^1}\binom{1}{0} & 0 & \cdots & 0 \\ \frac{1}{2^0}\left[\binom{2}{1}-\binom{2}{0}\right] & 0 & \frac{1}{2^2}\binom{2}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{2^N}\left[\binom{N}{\frac{N-1}{2}}-\binom{N}{\frac{N-3}{2}}\right] & 0 & \dots & \frac{1}{2^N}\binom{N}{0} \end{bmatrix}$$

for even N,

$$E=\begin{bmatrix} \frac{1}{2^0}\binom{0}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2^1}\binom{1}{0} & 0 & \cdots & 0 \\ \frac{1}{2^0}\left[\binom{2}{1}-\binom{2}{0}\right] & 0 & \frac{1}{2^2}\binom{2}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2^N}\left[\binom{N}{\frac{N}{2}}-\binom{N}{\frac{N-2}{2}}\right] & 0 & \frac{1}{2^N}\left[\binom{N}{\frac{N-2}{2}}-\binom{N}{\frac{N-4}{2}}\right] & \dots & \frac{1}{2^N}\binom{N}{0} \end{bmatrix}$$

$$T=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & N-1 & 0 \end{bmatrix}$$

$$Y(x)=[1 \quad x \quad \dots \quad x^N]^T$$

3. APPLICATION OF THE OPERATIONAL MATRIX METHOD

In this section, we implement the proposed method to solve Eq. (1) with Eq. (2). To apply the method, firstly, we find the operational matrices of terms of $g(x)$ in Eq. (1):

$$g(x)=FE^{-1}Y(x) \tag{14}$$

So, the residual form of the Eq. (1) can be introduced as:

$$\mathrm{RF_N(x)} = \left(\mathrm{C}\, E^{-1}\mathrm{Y(x)}\right)^{\mathrm{p}} - \sum_{q=0}^{n} f_q(x)\left(CE^{-1}T^{q}Y(x)\right) - FE^{-1}Y(x) \quad (15)$$

Eq. (15) includes coefficients c_i $i = 0,1,2,\dots,N$ is a non-linear equation for that c_i. To explore those c_i coefficients, apply the classical Tau method to the Eq. (15); that is, for $i = 0,1,2,\dots,N-2$

$$\langle RF_N(x), U_i(x)\rangle = \int_{-1}^{1} RF_N(x)U_i(x)dx = 0 \quad (16)$$

Eq. (16) generates $(N-1)$-times non-linear algebraic equations, while the numerical solution form Eq. (8) has $(N+1)$-times unbeknown coefficients. At this point, we have to take two initial conditions with the given Eq. (2), and so their contribution to the method is given by

$$u(0) \approx w_N(0) = CE^{-1}Y(0) = a_0 \quad (17)$$

$$u'(0) \approx w_N'(0) = CE^{-1}TY(0) = a_1 \quad (18)$$

$$\vdots$$

$$u^{(n-1)}(0) \approx w_N^{(n-1)}(0) = CE^{n-1}TY(0) = a_{n-1} \quad (19)$$

Hence, using Eqs. (16-19), we constitute $(N+1)$-times non-linear algebraic equations with $(N+1)$ unbeknown coefficients. The stack of Eqs. (16-19) can be solved using Newton's method for non-linear algebraic equation systems. This method derives from Newton's method, which takes an initial approximation for the solution and applies all component equations in a Taylor series. For more information, see [13].

3.1 Convergence and Error Bound

Chebyshev polynomials form a basis in $L^2[-1,1]$ and have a powerful advantage of localized properties in application sciences [13]. Also, $L^2[-1,1]$ is a Hilbert space. So, a function $w(x)$ which is continuous, is represented by the series

$$\mathrm{w(x)} = \sum_{\mathrm{k=0}}^{\mathrm{N}} \mathrm{c_i U_i(x)}$$

where

$$c_i = \frac{< \mathrm{w}(x), U_i(x) >}{< U_i(x), U_i(x) >}$$

Since a partial sum sequence $\tau_n = \sum_{k=0}^{N} c_i U_i(x)$ of the sequence is a Cauchy sequence in Hilbert space, the $\sum_{k=0}^{N} c_i U_i(x)$ series convergence to $w(x)$ [25]. The next theorem is a modified form in Ref. [26].

Theorem 3.1 Let u(x) ∈ $L^2[-1, 1]$. If $w_N(x)$ is defined by Eq. (9) is the approximation to u(x) and any reel number sequence $\{\lambda_i\}_{i=0}^{N}$ such that

$$\left||u(x) - w_N(x)\right|| \leq \left||u(x) - \sum_{i=0}^{N} \lambda_i U_i(x)\right|| \tag{20}$$

That is, the coefficient ci is the best choice for approximation to u(x).

Proof: If consider $w'_N(x) = \sum_{i=0}^{N} \lambda_i U_i(x)$, $\alpha(x) = u - w_N(x)$ and $\beta(x) = u - w'_N(x)$, we can write $\alpha(x) + w_N(x) = u(x) = \beta(x) + w'_N(x)$ and so

$$\beta(x) = w_N(x) - w'_N(x) + \alpha(x)$$

Now consider the inner product

$$< \alpha(x), w_N(x) - w(x) > = < u(x) - w_N(x), w_N(x) - w'_N(x) >$$

$$=< u - \sum_{i=0}^{N} \lambda_i U_i(x),\ \sum_{i=0}^{N} (c_i - \lambda_i) U_i(x) >$$

$$= \sum_{i=0}^{N} (c_i - \lambda_i) < u, U_i(x) > - \sum_{i,j=0}^{N} c_i \left(c_j - \lambda_j\right) < U_i(x), U_j(x) >$$

and since $\{U_i(x)\}_{i=0}^{N}$ is an orthogonal family

$$< \alpha(x), w_N(x) - w'_N(x) >= \sum_{i=0}^{N} (c_i - \lambda_i)[< u, U_i(x) > - c_i < U_i(x), U_i(x) >]$$

$$= \sum_{i=0}^{N} (c_i - \lambda_i) \left[< u, U_i(x) > - \frac{< u, U_i(x) >}{< U_i(x), U_i(x) >} < U_i(x), U_i(x) >\right] = 0$$

So, $\alpha(x)$ is orthogonal $w(x) - w'(x)$. Finally, the following equality

$$\left||\beta(x)\right||^2 = \left||w_N(x) - w'_N(x)\right||^2 + \left||\alpha(x)\right||^2$$

gives us

$$\left\|u(x)-\sum_{i=0}^{N}\lambda_i U_i(x)\right\|^2=\left\|\sum_{i=0}^{N}(c_i-\lambda_i)U_i(x)\right\|^2+\left||u(x)-\sum_{i=0}^{N}c_i U_i(x)|\right|^2$$

and

$$\left\|u(x)-\sum_{i=0}^{N}c_i U_i(x)\right\| \leq \left\|u(x)-\sum_{i=0}^{N}\lambda_i U_i(x)\right\|$$

Hence, proof is done.

Theorem 3.2 If $u(x)$ has n-times continuous derivatives in [−1, 1] and $w_N(x)$ is the best square approximation of $u(x)$, then the error bound for approximation $w_N(x)$ is as follows:

$$||u(x)-w_N(x)|| \leq \frac{MS^{N+1}}{(N+1)!}\sqrt{\frac{\pi}{2}} \tag{21}$$

where $M=max\left(w_N^{(N+1)}(x)\right)$ and $S=max\{x-x_0,x_0\}$.

Proof: The function u(x) can be expanded by Taylor expansion as

$$u(x)=u(x_0)+(x-x_0)u'(x_0)+\cdots+\frac{(x-x_0)}{N!}u^{(N)}(x_0)$$
$$+\ \frac{(x-x_0)}{(N+1)!}u^{(N+1)}(\xi)$$

where $x_0 \in$ [−1, 1] and $\xi \in [x_0$, x) or $\xi \in$ (x, x_0].

Assume that $w_N(x)$ is a Taylor polynomial is an approximation for u(x) and so,

$$w_N^*(x)=u(x_0)+(x-x_0)u'(x_0)+\frac{(x-x_0)}{2!}u''(x_0)+\cdots+\frac{(x-x_0)}{N!}u^{(N)}(x_0)$$

then

$$||u(x)-w_N^*(x)||=\left|\frac{(x-x_0)^{(N+1)}}{(N+1)!}u^{(N+1)}(\xi)\right|$$

We can write the following inequality for the approximation as

$$||u(x)-w_N(x)||^2 \leq ||u(x)-w_N^*(x)||^2 \leq \int_{-1}^{1}w(x)||u(x)-w_N^*(x)||^2dx$$

$$\leq \int_{-1}^{1} w(x)\left[\frac{(x-x_0)^{(N+1)}}{(N+1)!}u^{(N+1)}(\xi)\right]^2 dx \leq \frac{M^2S^{2N+2}}{[(N+1)!]^2}\frac{\pi}{2}$$

Finally, by taking the square roots of both sides, we have

$$\left|\left|u(x)-w_N(x)\right|\right| \leq \frac{MS^{N+1}}{(N+1)!}\sqrt{\frac{\pi}{2}} \tag{22}$$

Moreover, if the approximate solution $w_N(x)$ is replaced in Eq. (1), we expect the Eq. (1) to be nearly ensured from [31], that is;

$$\left|w_N^p(x)-\sum_{q=0}^{n} f_q(x)w_N^{(q)}(x)-g(x)\right| \approx 0$$

which is called the error estimation function (EEF) of the numerical solution. That is, EEF is

$$EEF_N(x) = w_N^p(x) - \sum_{q=0}^{n} f_q(x)w_N^{(q)}(x) - g(x) \tag{23}$$

4. NUMERICAL APPLICATIONS

In this section, we demonstrate the validity and applicability of the proposed method. Our numerical results are compared to the exact solution and existing methods. All numerical schemes are run in Maple. The absolute error for point x_i is

as follows

$$AE_N(x_i) = |u(x_i) - w_N(x_i)| \tag{24}$$

Example 4.1 For the first numerical application, consider the following second-order non-linear differential equation:

$$u''(x) - xu'(x) + u(x) + u^2(x) = f(x) \tag{25}$$

and initial conditions

$$u(0) = 0, u'(0) = 1 \tag{26}$$

If $f(x) = sin^2x - xcosx$, the exact solution is $u(x) = sinx$. One can want to investigate the eight terms Chebyshev series solution for numerical results of Eqs. (25-26):

$$w_7(x) = \sum_{k=0}^{7} c_i U_i(x) \tag{27}$$

We implement the operational matrix method to Eqs. (25-26) to reveal the coefficients c_i. For $N = 7$, our operational matrices are

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 1/8 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & 3/16 & 0 & 1/16 & 0 & 0 & 0 \\ 0 & 5/32 & 0 & 1/8 & 0 & 1/32 & 0 & 0 \\ 5/64 & 0 & 9/64 & 0 & 5/64 & 0 & 1/64 & 0 \\ 0 & 7/64 & 0 & 7/64 & 0 & 3/64 & 0 & 1/128 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

and

$$Y(x) = [1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad x^5 \quad x^6 \quad x^7]^T$$

$$C = [c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad c_6 \quad c_7]$$

If those matrices are put into Eq.(15), we generate the following residual matrix form:

$$RF_N(x) \approx CE^{-1}T^2Y(x) - xCE^{-1}TY(x) + CEY(x) + (CE^{-1}Y(x))^2 \tag{28}$$

and the initial conditions of Eq.(26) can be expressed

$$u(0) \cong w_7(0) = CE^{-1}Y(0) = c_0 - c_2 + c_4 - c_6 = 0 \tag{29}$$

$$u'^{(0)} \cong w_7'(0) = CE^{-1}TY(0) = 2c_1 + 4c_3 + 6c_5 - 8c_7 = 1 \tag{30}$$

Applying our numerical method which, is mentioned in Section 3, eight non-linear algebraic equation systems are obtained with Eq. (28) - Eq. (30). If these systems are figured out by Maple, the polynomial approximation can be found by

$$w_7(x) = 1 - 0.166663912x^2 + 0.00832604x^5 - 0.00019083395x^7$$

For Eq. (25), we applied our numerical procedure for values N = 6, 7, 9 and obtained numerical results in Table 1 and Figures 1-2. In Table 1, obtained numerical solutions and their absolute errors are tabulated for some values of x. Figure 1 plots the second kind of Chebyshev polynomial and exact solution. In Figure 2, absolute error functions AE_N are displayed to compare the quality of approximations. Figure 2 shows us that errors in the neighbourhood of the initial point are lower than other points. Moreover, theoretical error bounds are 10^{-4}, 10^{-6} and 10^{-8} for $N = 6,7,9$, respectively.

Table 1. Numerical results of Example 4.1 for different N

		Present Method					
x	Exact Solution	N = 6	AE_6	N = 7	AE_7	N = 9	AE_9
-1.0	-0.841470	-0.841552	0.818E-4	-0.841470	0.361E-6	-0.841470	0.124E-8
-0.8	-0.717356	-0.717421	0.655E-4	-0.717356	0.262E-6	-0.717356	0.886E-9
-0.6	-0.564685	-0.564685	0.426E-4	-0.564642	0.217E-6	-0.564642	0.700E-9
-0.4	-0.389418	-0.389435	0.168E-4	-0.389418	0.113E-6	-0.389418	0.463E-9
-0.2	-0.198669	-0.198671	0.242E-5	-0.198669	0.196E-7	-0.198669	0.957E-10
0.0	0.000E-0	0.000E-0	0.000E-0	0.000E-0	0.000E-0	0.000E-0	0.000E-0
0.2	0.198669	0.198671	0.262E-5	0.198669	0.199E-7	0.198669	0.968E-10
0.4	0.389418	0.389435	0.171E-4	0.389418	0.112E-6	0.389418	0.459E-9
0.6	0.564642	0.564683	0.413E-4	0.564642	0.207E-6	0.564642	0.659E-9
0.8	0.717356	0.717414	0.584E-4	0.717356	0.222E-6	0.717356	0.741E-9
1.0	0.841470	0.841531	0.609E-4	0.841470	0.257E-6	0.841470	0.885E-9

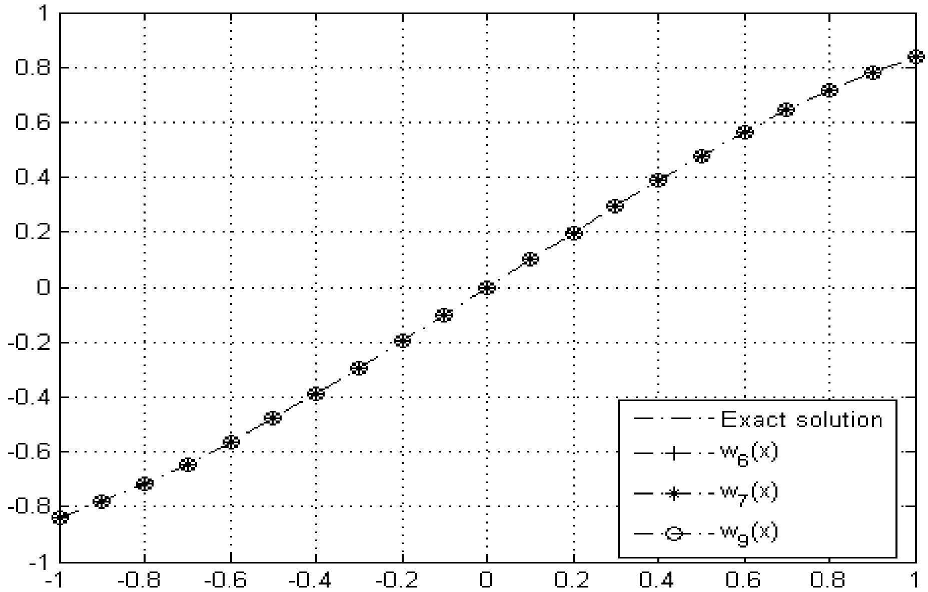

Figure 1. Comparison of the approximate results for N = 6, 7, 9.

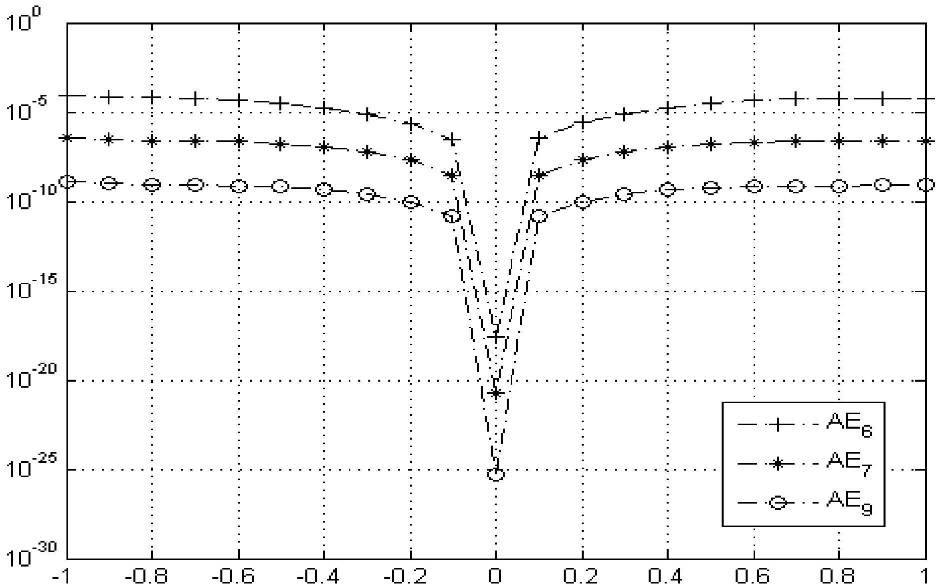

Figure 2. Comparison of the absolute errors for N = 6, 7, 9.

Example 4.2 In this example, we take the following Duffing equation [3,28,29,30,32]

$$8u^3(x) = -u''(x) - 2u'(x) - u(x) + e^{-3x}, u(0) = 0.5, u'(0) = -0.5 \quad (31)$$

The Duffing equation is both a linear and non-linear second-order differential equation which describes the motion of a damped, driven oscillator with a non-linear restoring force. It is used in studying chaotic, mechanics and electrical systems, where the restoring force is not proportional to displacement. In general, the Duffing Equation can be given by the following equation [38]

$$u''(x) + au'(x) + bu(x) + cu^3 = f(x) \tag{32}$$

where u is the system's displacement as a function of time, u' is the velocity of the system and u'' is the system's acceleration.
The problem (31) is a linear case of (32). Problem (31) has been numerically solved using the Boole collocation method (BCM), the Adoman decomposition method (ADM) and the Laguerre wavelet method (LWM). All results are presented and compared in Table 2 and Figure 3.

Table 2. Numerical results and comparison of Example 4.2 for different methods

x	*Exact Solution*	*BCM*	*ADM*	*LWM*	*COM (N=6)*	*COM (N=9)*
0.0	*0.000E-0*	*0.000E-0*	*0.000E-0*	*0.000E-0*	*0.000E-0*	*0.000E-0*
0.2	*0.2.8618*	*2.8618E-7*	*5.47E-7*	*1.08E-9*	*2.3088E-8*	*9.5670E-10*
0.4	*0.389418*	*4.7533E-7*	*6.29E-5*	*5.13E-9*	*2.7245E-6*	*1.2442E-09*
0.6	*0.564642*	*6.6969E-7*	*9.86E-4*	*7.99E-10*	*7.6771E-6*	*3.6024E-10*
0.8	*0.717356*	*1.2269E-6*	*9.93E-3*	*5.08E-11*	*9.1013E-6*	*1.3090E-10*
1.0	*0.841470*	*3.1614E-5*	*3.18E-2*	*1.58E-6*	*5.7648E-6*	*1.7343E-10*

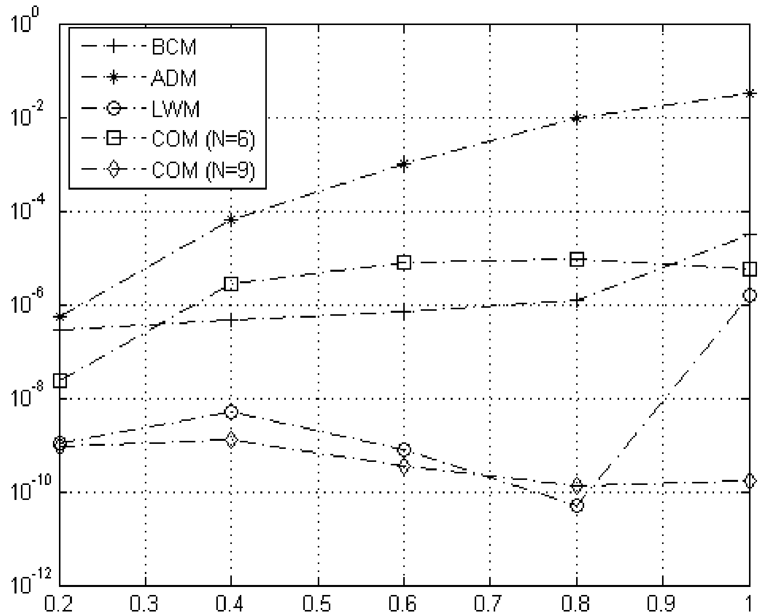

Figure 3. Comparison of the approximate results of different methods

Example 4.3 Now, consider the following third-order differential equations with two non-linear terms:

$$u^3(x) + u^2(x) = -u'''(x) + u''(x) + 2u'(x) + u(x) + e^{3x} + e^{2x} + e^x \quad (33)$$

with initial conditions $u(0) = 1, u'(0) = 1, u''(0) = 1$. $u(x) = e^x$ is the exact solution of the Eq. (33) with initial conditions. This problem has $u^3(x)$ and $u^2(x)$ non-linear terms. Our numerical scheme is adapted for this example. Hence, all results are presented in Table 3 for various N. In this example, theoretical errors are 10^{-2}, 10^{-3},and 10^{-4}, for $N = 4,5,6$, respectively.

Table 3. Numerical results of Example 4.3 for different N

		Present Method					
x	Exact Solution	N = 6	AE_6	N = 5	AE_5	N = 4	AE_4
-1.0	0.368794	0.368010	1.310E-4	0.369386	1.507E-3	0.368908	1.109E-2
-0.8	0.449328	0.449448	1.195E-4	0.450082	7.539E-4	0.448984	3.447E-3
-0.6	0.548811	0.548889	7.781E-5	0.549072	2.609E-4	0.548396	4.155E-3
-0.4	0.670320	0.670350	3.093E-5	0.670368	4.860E-5	0.670148	1.717E-3
-0.2	0.818730	0.818735	4.640E-6	0.818732	1.415E-6	0.818707	2.305E-4
0.0	1.000000	1.000000	0.000E-0	1.000000	0.000E-0	1.000000	0.000E-0
0.2	1.221402	1.221397	5.075E-6	1.221412	9.260E-6	1.221414	1.133E-5
0.4	1.491824	1.491787	3.711E-5	1.491938	1.136E-4	1.491799	2.481E-4
0.6	1.822118	1.822015	1.028E-4	1.822606	4.878E-4	1.821468	6.501E-3
0.8	2.225540	2.225366	1.740E-4	2.226829	1.288E-3	2.222193	3.334E-2
1.0	2.718281	2.718083	1.980E-4	2.720730	2.448E-3	2.707208	1.107E-2

Example 4.4 In this example, we consider the following non-linear differential equation [3]:

$$u^2(x) = 1 - u'(x) \quad (34)$$

with initial conditions $u(0) = 0$. Its solution is $u(x) = (e^{2x} - 1)/(e^{2x} + 1)$. Gülsu et al. [34] investigated the numerical solutions using the Taylor matrix method (TMM). They obtained fifth-degree Taylor polynomial approximation by their proposed method:

$$u(x) = x - \frac{x^3}{3} + \frac{2x^5}{3}$$

For $N = 5$, our result is

$$w_5(x) = 0.99628434 - 0.30394114x^3 + 0.06925218x^5$$

Both results are plotted in Figure 4 to compare TMM and COM. While both approximations are seen in the same line in Figure 4, Figure 5 shows the difference between both approximations. Our numerical scheme has nearly absolute errors of 10^{-4} in the interval $[-1,1]$. However, TMM only has a good approximation in the neighbourhood at the initial point.

Example 4.5 Consider following non-linear differential equation

$$u^2(x) = u''(x) - xu'(x) + 4xe^x - x^3e^x + x^2e^{2x} - 2x^3e^{2x} + x^4e^{2x} \quad (35)$$

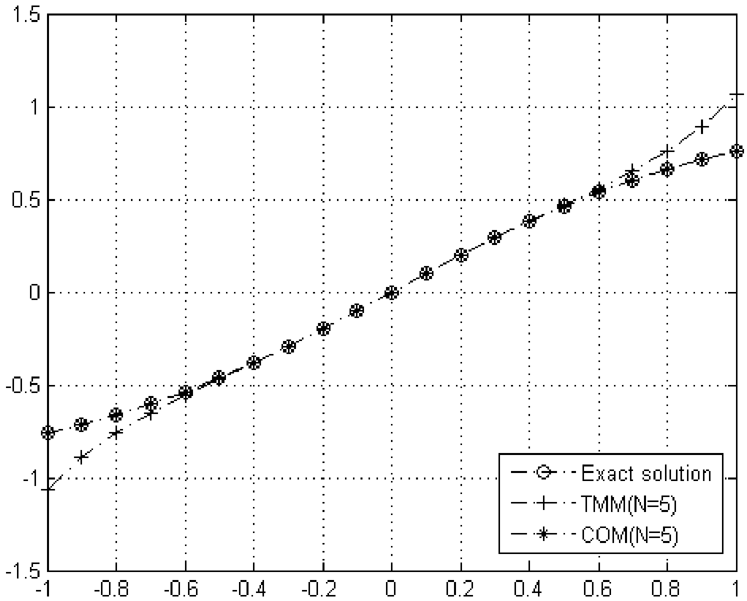

Figure 4. Comparison of the approximate results for Ex. 4.4

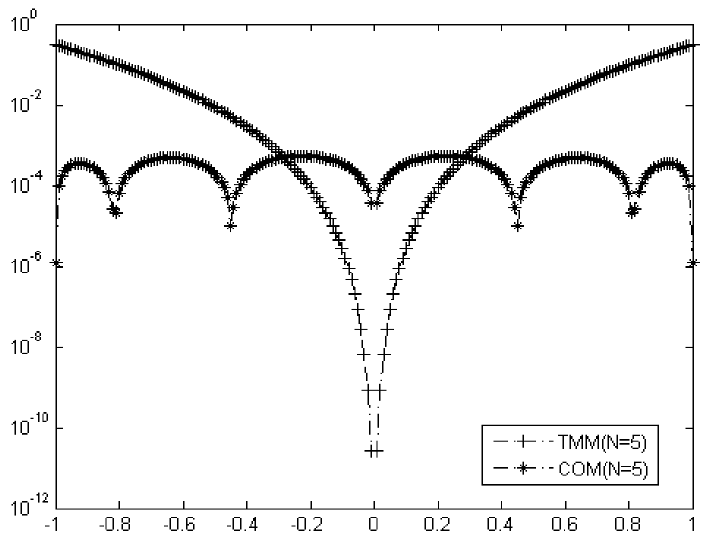

Figure 5. Comparison of the absolute errors for Ex. 4.4

with initial conditions $u(0) = 0, u'(0) = 1$ and exact solution $u(x) = x(1-x)e^{x}$. The Chebyshev operational solutions $w_N(x)$ and error estimation function EEF_N of problem 35 are investigated for $N = 5, 8$ and 11. Figure 6 and Figure 7 show the absolute errors and error estimation functions, respectively. Two comparisons say that the exact solution and numerical solutions are compatible with the exact solution.

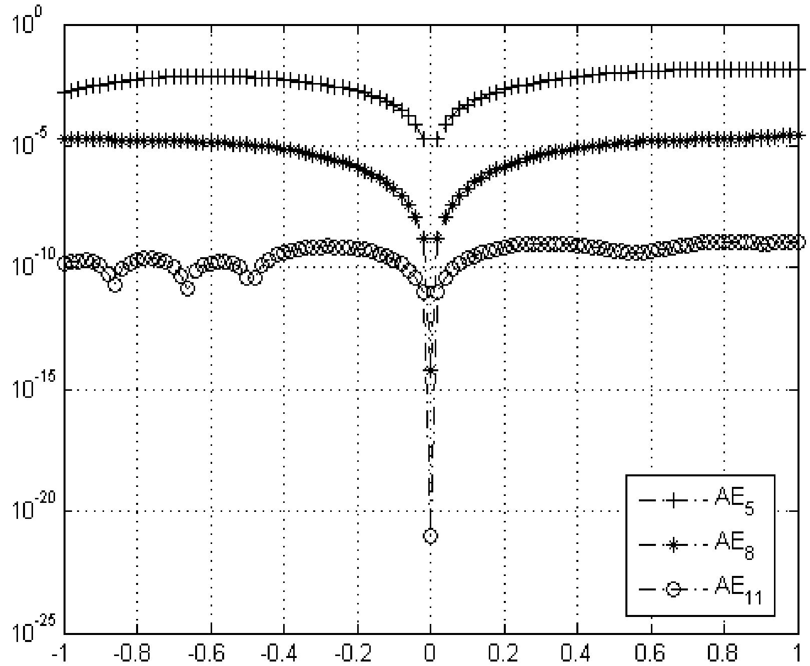

Figure 6. Comparison of the approximate results of Example 4.5

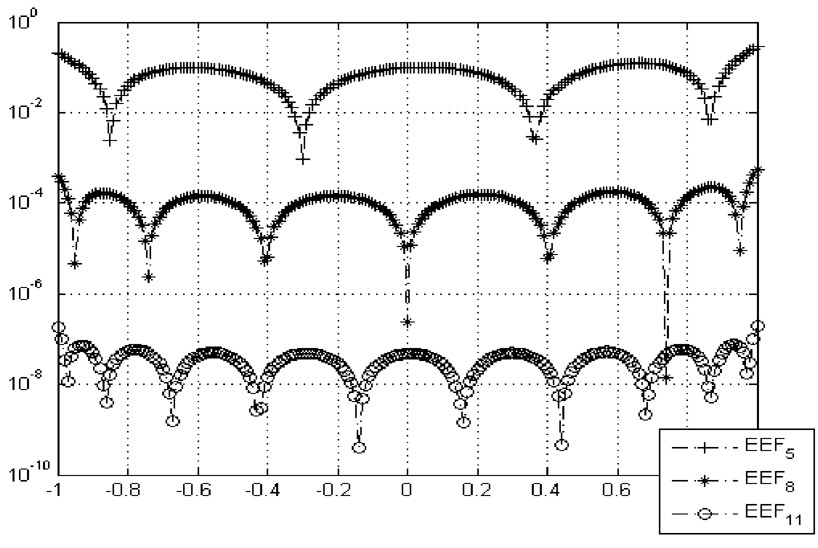

Figure 7. Comparison of the error estimation function of Example 4.5.

Example 4.6 Let us take the Riccati differential equation [11, 35]

$$u^2(x) = -u'(x) + 2u(x) + 1, u(0) = 0 \qquad (36)$$

with the exact solution

$$u(x) = 1 + \sqrt{2}\, tanh\left[1 + \sqrt{2}\, x + \tfrac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

The Riccati equation arises in applied sciences such as optimal control, diffusion and random processes [37]. Eq. (36) has been figured out by the Chebyshev series method [11], the general Jacobi matrix method [35] and the homotopy perturbation method [36]. The approximate polynomial solutions of the proposed methods (COM) came through the special numerical scheme from Table 4, with small N values. Also, AE and EEF are compared in Fig.8. All results have the same harmony regarding the goodness of correction.

Table 4. Numerical results and comparison for Example 4.6.

x	Exact Solution	Method of [35]	Method of [36]	Method of [11]	COM(N=5)	COM(N=8)
0.1	0.11029	2.00E-4	2.50E-3	5.00E-2	1.276E-3	2.702E-4
0.2	0.24197	1.00E-3	1.80E-3	2.00E-2	1.161E-3	5.018E-4
0.3	0.39510	1.60E-3	1.60E-3	4.51E-2	7.237E-4	5.221E-4
0.4	0.56781	1.00E-3	1.80E-3	8.06E-2	3.986E-3	3.104E-4
0.5	0.75601	2.00E-3	2.10E-3	12.7E-2	7.362E-3	6.073E-5
0.6	0.95356	3.00E-3	2.20E-3	19.0E-2	9.069E-3	5.122E-5
0.7	1.15294	9.00E-4	2.30E-3	27.4E-3	7.682E-3	3.469E-4
0.8	1.34636	5.20E-3	2.10E-3	38.1E-2	3.423E-3	5.835E-4
0.9	1.52691	1.30E-3	2.00E-3	56.0E-2	4.679E-4	2.876E-4

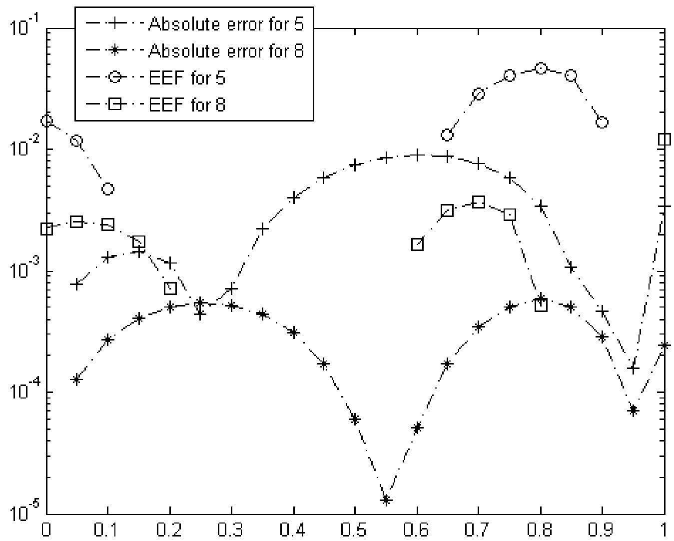

Figure 8. Comparison of AE and EEF for Example 4.6

Example 4.7 Consider the Duffing's equation [32]

$$u''(x) + u'(x) + u(x) + u^3(x) = cos^3x - sinx \tag{37}$$

with initial conditions $u(0) = 1, u'(0) = 0$ and exact solution $u(x) = \cos(x)$. The Chebyshev operational matrix method solution and Adomian's decomposition method solution is tabulated in Table 5. Numerical comparison of obtained results shows the efficiency of the proposed method.

Table 5. Numerical results and comparison for Example 4.7.

x	A DM [32]	COM (N=8)	COM (N=10)
0.0	0.0000E-0	0.0000E-0	0.0000E-0
0.1	6.5259E-11	5.9581E-10	1.6930E-12
0.2	1.4309E-8	1.9593E-9	5.0219E-12
0.3	3.6064E-7	3.0890E-9	6.4435E-12
0.4	3.8877E-6	3.0883E-9	4.2331E-12
0.5	2.7689E-5	1.8344E-9	5.4050E-13
0.6	1.5755E-4	2.3267E-10	6.9586E-13
0.7	7.7543E-4	4.4160E-10	1.3120E-12
0.8	3.3925E-3	1.9163E-10	2.3364E-12
0.9	1.3318E-3	7.2342E-10	6.2245E-15
1.0	4.7237E-2	1.7018E-11	8.2272E-14

Example 4.8 Let us consider the Van der Pol equation [11]

$$u''(x) - 0.05(1 - u^2)u'(x) + u(x) = 0 \tag{38}$$

with initial conditions $u(0) = 0, u'(0) = 0{,}5$. The Van der Pol equation is a second- order non-linear differential equation commonly used to model oscillatory systems with damping and nonlinearity. The problem Eq. (38) does not have an analytical solution, or its solution is unknown. However, a fourth-fifth order Runga-Kutta method, Gear single-step extrapolation method, Chebyshev series method, and our method are proposed to compare the obtained numerical results in Table 6.

Example 4.9 Finally, we consider the following Lane-Emden equation

$$u''(x) + \frac{\alpha}{x}u'(x) + u^n(x) = 0 \tag{39}$$

$u(0) = 1, u'(0) = 0$ are initial conditions.

Table 6. Numerical results and comparison for Example 4.8.

x	Runga-Kutta method	Gear method	Chebyshev series method	Our method
0.0	0.0	0.0	0.0	0.0
0.1	0.0500416616	0.0500416561	0.0500416561	0.500416561
0.2	0.0998321792	0.0998321668	0.0998321663	0.0998321667
0.3	0.1488697072	0.1488697063	0.1488697052	0.1488697061
0.4	0.1966564313	0.1966564167	0.1966564148	0.1966564150
0.5	0.2427036961	0.2427036728	0.2427036829	0.2427036763
0.6	0.2865373842	0.2865373709	0.2865373683	0.2865373717
0.7	0.3277032041	0.3277031577	0.3277031549	0.3277031562
0.8	0.3657715349	0.3657715117	0.3657715085	0.3657715004
0.9	0.4003426519	0.4003425836	0.4003425800	0.4003425843
1.0	0.4310507546	0.4310507104	0.4310507066	0.4310506934

The Lane-Emden equation is a second- order ordinary differential equation used in astrophysics to describe the structure of a polytropic star, which is a star whose pressure is related to its density by a polytropic equation of state. It models the distribution of mass and pressure within such a star. The general form of the Lane-Emden equation is [39, 40, 41, 42, 43,44,45]

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{du}{dx}\right)+u^n=0 \tag{40}$$

where x is a dimensionless radial coordinate, u(ξ) is a dimensionless form of density, n is the polytropic index and α is the geometry of the gas vessel. Moreover, $u(0)=1$ and $u'(0)=0$ are conditions. They mean central density is maximum and maintain regularity, respectively. The polytropic index n has an important role in the solution of this equation. The residuel form of Eq. (39) is

$$RF_N(x)\approx CE^{-1}T^2Y(x)+\frac{\alpha}{x}CE^{-1}TY(x)+CE^{-1}Y(x)+(CE^{-1}Y)^n$$

For $\alpha=2$ and $n=1$, Tables 7 and 8 show the comparison of numerical solution and error of the advanced variational iteration method (AVIM), advanced Adomian decomposition method (AADM), Monte Carlo method (MC) and Chebyshev neural network method (CNN). We compare the numerical solution of several numerical methods and the exact solution in Fig. 9. Also, a comparison of absolute error results is shown in Fig.10. We understand that compared to several methods and our method, the absolute errors of our method are smaller than others.

Table 7. Comparison of numerical results of Example 4.9.

x	*Exact solution*	*Our method*	*AVIM [41]*	*AADM [46]*	*MC [47]*	*CNN [48]*
0.1	*0.998334*	*0.998334*	*0.998334*	*0.998334*	*0.998300*	*1.001800*
0.2	*0.993346*	*0.993346*	*0.993346*	*0.993347*	*0.993280*	*0.990500*
0.3	*0.985067*	*0.993346*	*0.993346*	*0.985068*	*0.984968*	*0.983900*
0.4	*0.973545*	*0.973545*	*0.973545*	*0.973547*	*0.973415*	*0.973400*
0.5	*0.958851*	*0.958851*	*0.958851*	*0.958689*	*0.958689*	*0.959800*
0.6	*0.941070*	*0.941070*	*0.941070*	*0.940879*	*0.940879*	*0.941700*
0.7	*0.920310*	*0.920310*	*0.920310*	*0.920090*	*0.922560*	*0.921000*
0.8	*0.896695*	*0.896695*	*0.896695*	*0.896447*	*0.900190*	*0.892500*
0.9	*0.870363*	*0.870363*	*0.870368*	*0.870090*	*0.875440*	*0.870000*
1.0	*0.841470*	*0.841470*	*0.841471*	*0.841175*	*0.848680*	*0.843000*

Table 8. Comparison of absolute errors of Example 4.9

x	Our method	AVIM [41]	AADM [46]	MC [47]	CNN [48]
0.1	7.371719E-19	1.1022E-16	1.98385E-10	3.417E-5	3.466E-3
0.2	2.043242E-18	0.0000000	1.26914E-8	6.665E-5	2.847E-3
0.3	2.477612E-18	1.11022E-16	1.44462E-7	9.936E-5	1.167E-3
0.4	2.020566E-18	2.77556E-15	8.10895E-7	1.308E-4	1.458E-4
0.5	1.788870E-18	3.90799E-14	3.08946E-6	1.620E-4	9.489E-4
0.6	2.039832E-18	3.49165E-13	9.21101E-6	1.918E-4	6.292E-4
0.7	1.999435E-18	2.21756E-12	2.31849E-5	2.210E-4	6.890E-4
0.8	1.801515E-18	1.10021E-11	5.15530E-5	2.481E-4	4.195E-3
0.9	1.872516E-18	4.51812R-11	1.04267E-4	2.732E-4	3.632E-4
1.0	1.772793E-18	1.59828E-10	1.95682E-4	2.959E-4	1.529E-2

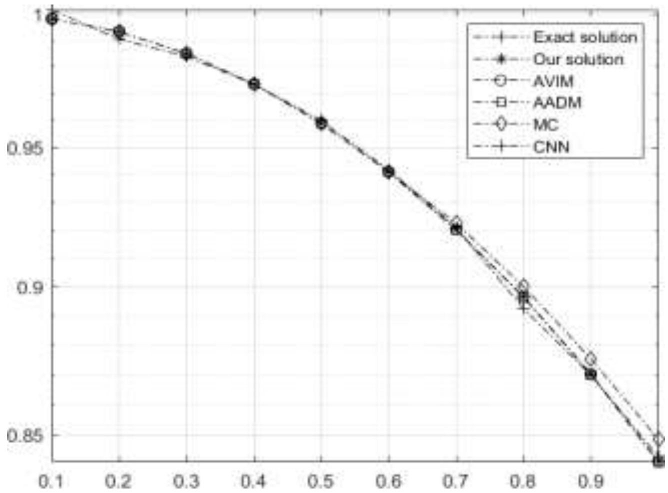

Figure 9. Numerical result of Example 4.9

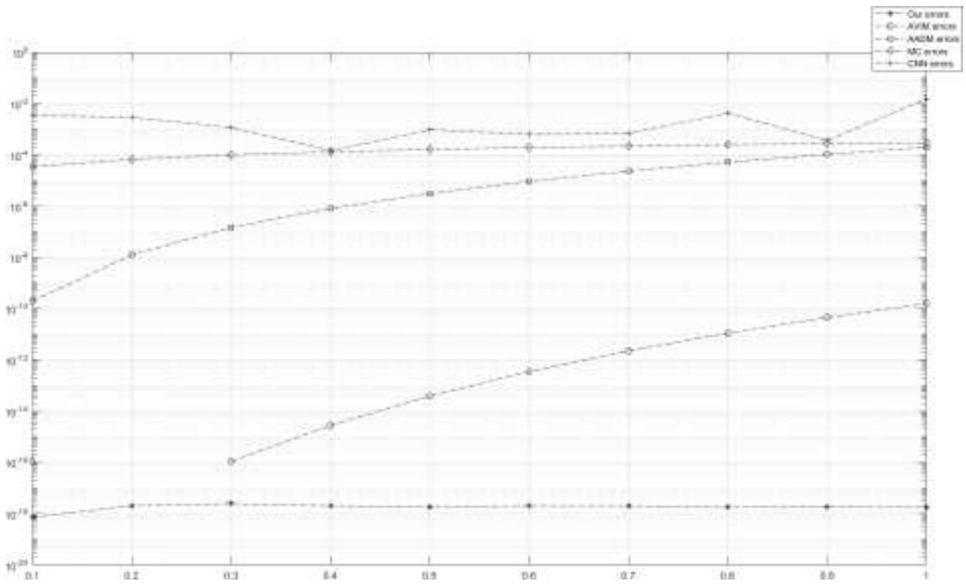

Figure 10. Absolute error of Example 4.9.

5. CONCLUSION

This paper considers a Chebyshev operational matrix method to determine a class non-linear differential equation such as Riccati equation, Duffing equation, Van der Pol equation and Lane-Emden equation. The operational matrix method and Chebyshev polynomial are the main keys of the proposed method to transform the given nonlinear differential equation to a non-linear algebraic system, including unknown coefficients of the Chebyshev series. All computations are run using Maple mathematical software. Our numerical scheme has some advantages:

- While the absolute error is deficient, the effort of the numerical scheme is very low.

- From Theorem 1, the approximation coefficients are the best choice for such a numerical scheme.

- Our theoretical errors and absolute errors are consistent.

- Our method gives lower absolute errors than others.

Additionally, we give numerical comparison between the proposed method and other existing methods such as the Jacobi matrix method, homotopy perturbation method, Adomian decomposition method, Laguerre wavelet method and Boole collocation method. The comparison results from the tables and figures favour our numerical scheme. Lastly, note that the numerical scheme may be adapted to Fredholm or Volterra integral equations, integro-differential equations, differential-difference equations etc.

Data Availability Statement No Data associated in the manuscript

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